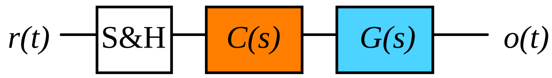
Effects of sampling: details

Consider the following system:



We haven't done anything digital here yet, just put a sample and hold upstream of two analog subsystem blocks, *G(s)* and *C(s)*. All signals at all points are analog. Just to the right of the S&H, you can describe the stair-stepped sample&hold approximation of *r(t)* as

$$\bar{r}(t) = r(0)\big[u(t) - u(t-T)\big] + r(T)\big[u(t-T) - u(t-2T)\big] + \ldots r(nT)\big[u(t-nT) - u(t-[n+1]T)\big]\ldots$$

where T is the sampling period and u is the unit step function.

This analytic, but messy, function of time has a LaPlace Transform,

$$\bar{R}(s) = r(0)\left[\frac{1}{s} - \frac{1}{s}e^{-sT}\right] + r(T)\left[\frac{1}{s}e^{-sT} - \frac{1}{s}e^{-2sT}\right] + \ldots\ldots = \frac{1-e^{-sT}}{s}\underbrace{\sum_{n=0}^{\infty} r(nT)e^{-nsT}}_{R^*(s)} .$$

The transform *R*(s)* (the sum in the above equation) is the LaPlace transform of a train of pulses, separated in time by *T*, whose heights are the values of the sampled sequence, *r(nT)*.

Now, compare this result to the z-transform of the same sequence,

$$R^*(s) = \sum_{n=0}^{\infty} r(nT)e^{-nsT} \quad \text{and} \quad R(z) = \sum_{n=0}^{\infty} r(nT)z^{-n} .$$

The LaPlace transform and the z-transform correspond exactly, if $z = e^{sT}$.

To study the entirely analog (but also sampled) system in the diagram, you must work in the z-domain because of the non-linear nature of the sample and hold operation (which you can see above by the presence of non-polynomial terms like $\exp\{sT\}$). The transfer function of this system in the z-domain is

$$H(z) = \Im\left\{\frac{1-e^{-sT}}{s}C(s)G(s)\right\} = (1-z^{-1})\Im\left\{\frac{C(s)G(s)}{s}\right\} = \frac{z-1}{z}\Im\left\{\frac{C(s)G(s)}{s}\right\} .$$

In this equation, $\Im\{F(s)\}$ means the function of z "equivalent to" *F(s)*. You can see from these results that is isn't enough just to find the z-transform that is equivalent to the

LaPlace transform of a given system transfer function. You must explicitly take account of the sample-and-hold. Its presence is real.

In both the s- and the z- domains, the transfer function of a system is the transform (Laplace or z-) of the system's impulse response. For continuous systems, this response is a continuous function of time, while for discrete systems it is a sequence of samples of a continuous function which is the "underlying" impulse response.. Thus, expanding *F(s)* into the residue-pole form shows that the impulse response of any continuous linear system is a sum of exponential functions of time: if

$$H(s) = \sum_i \frac{A_i}{s - p_i}, \text{ then } h(t) = \sum_i A_i e^{pt} ,$$

where the sum runs over all the poles. In the z-domain, we have already shown that the z-transform of an exponential time sequence is given by the z-transform pair,

$$Ae^{-anT} \overset{Z}{\Leftrightarrow} A\frac{z}{z - e^{-aT}} .$$

Now, compare the LaPlace and the z-transforms for an exponential function or sequence:

$$\frac{A}{s+a} \Leftrightarrow Ae^{-at} \overset{Z}{\Leftrightarrow} A\frac{z}{z - e^{-aT}} .$$

Since the time responses of both continuous and sampled systems can be expressed as a sum of exponential time functions (continuous) or sequences (sampled), this comparison suggests the following procedure for finding $\Im\{F(s)\}$:

1. Put *F(s)* into residue-pole form, using a partial fraction expansion, or some equivalent approach.
2. For each term in the residue-pole sum, write the equivalent term in the z-domain using the known values for the residue, *A*, and the pole, *-a*.
3. Sum the terms in the z-domain to get a rational polynomial in z.

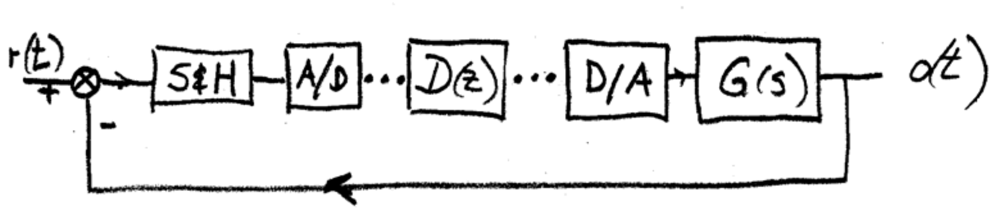
Doing this procedure by hand is very tedious and error-prone. Fortunately, a numerical algorithm is available.

Another interesting (and potentially dangerous) property of z-transforms is that they can't be multiplied for cascaded systems.

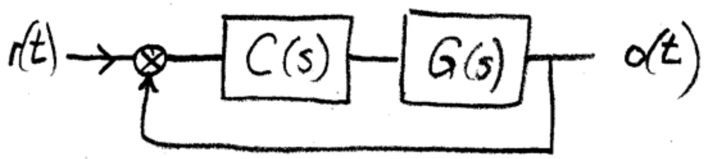
Sampling speed is usually at a premium in sampled-data control systems that use computers in the loop. Having the computer sample both the command signal, *r(t)*, and the feedback signal cuts the available sampling frequency in half. Consequently, most single-input-single-output (SISO) computer-based feedback control systems are of the unity-feedback type, where the feedback transfer function is unity and the sampling is performed after the command and the feedback signals are differenced. For such a system with a digital cascade controller, *D(z)*, the transfer function is

$$H(z) = \frac{D(z)\Im\left\{\frac{G(s)}{s}\right\}}{1 + D(z)\Im\left\{\frac{G(s)}{s}\right\}} .$$

The system block diagram in this case is



For designers with some insights what sort of cascade compensator, *C(s),* might work in the system below,



it is often convenient to start with the digital equivalent of *C(s),* and then reduce the sampling frequency while making modifications in *C(s)* to maintain system performance.

Caveats:

1) Even if the “equivalence” condition,

$$D(z) = \frac{\Im\left\{\dfrac{C(s)G(s)}{s}\right\}}{\Im\left\{\dfrac{G(s)}{s}\right\}}$$

is satisfied, the two unity feedback systems pictured above perform alike only at high sampling frequencies.

2) To find digital equivalents of cascaded continuous systems, multiply the s-domain transfer functions first, THEN transform to the z-domain.