Side-by-side comparison of LaPlace and z- transforms

Since we want to learn to design continuous and sampled control systems simultaneously, it is useful to emphasize how the mathematical transforms we use (LaPlace for continuous systems and z-transforms for sampled systems) are analogous to each other.

1. **The system model**: This is the equation that relates the output to the input and describes how the system operates.

   Continuous system: The model for a continuous linear system is a linear *differential* equation with constant coefficients, and has the general form $\alpha_0 o(t) + \alpha_1 \frac{do}{dt} ... + \alpha_M \frac{d^M o}{dt^M} = \beta_0 i(t) + \beta_1 \frac{di}{dt} ... + \beta_N \frac{d^N i}{dt^M}$, where the output *o(t)* and the input *i(t)* are continuous functions of time.

   Sampled system: The model for a sampled (discrete) linear system is a linear *difference* equation with constant coefficients, and has the general form,

   $$a_0 o_n + a_1 o_{n-1} + ... a_M o_{n-M} = b_0 o_n + b_1 o_{n-1} + ... b_N o_{n-N} \,,$$

   where the output and input are sequences of values, the *k*-th member of the sequence representing the value of the output or the input *sampled* at time $t = kT$. Here *T* is the sampling period, which is the reciprocal of the sampling frequency. Thus, for example, output *n*-1 occurred *T* earlier in time than output *n*. One limitation of a discrete model is that it contains no information about the value of any function at times between samples.

2. **Operational definition of the transform**: This is the operation you perform to go from the system model equation to its transform. You want to do this transformation because it transforms a differential or difference equation into a simpler algebraic equation.

   Continuous system: To transform the differential equation, replace every occurrence of *m*-th differentiation by the product of transform and the *m*-th power of the transform variable, *s*,

   $$\frac{d^m o}{dt^m} \rightarrow s^m O(s) \text{ and } \frac{d^m i}{dt^m} \rightarrow s^m I(s)$$

   where *O(s)* and *I(s)* are the LaPlace transforms of the time functions *o(t)* and *i(t).*

   Sampled system: To transform the difference equation, replace every time shift to *p* samples earlier in the sequence by the product of z^{-p} and the transform of the sequence as

   $$o_{n-p} \rightarrow z^{-p} O(z) \quad \text{and } i_{n-p} \rightarrow z^{-p} I(z)$$

   where *O(z)* and *I(z)* are the z-transforms of the time sequences, o_k and i_k.

3. **Transfer function:** Once you have made the transformation of the differential or difference equation using the operational definition above, you can always solve for the transfer function of the system in rational polynomial form.

   Continuous system:

   $$H(s) \equiv \frac{O(s)}{I(s)} = \frac{\beta_0 + \beta_1 s + ... \beta_N s^N}{\alpha_0 + \alpha_1 s + ... \alpha_M s^M}$$

For a continuous system, solving the transformed differential equation for the transfer function gives a rational polynomial without any further work.
Sampled system: In the sampled case, solving the transformed system model equation leads directly to

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_N z^{-N}}{a_0 + a_1 z^{-1} + \ldots + a_M z^{-M}},$$

which is not a rational polynomial in *z*. However, you can easily turn it into a rational polynomial by determining the larger of *N* or *M*, and then multiplying numerator and denominator by *z* raised to that power. In the case that *M*>*N*, you get

$$H(z) = \frac{b_N z^{M-N} + \ldots b_1 z^{M-1} + b_0 z^M}{a_M + \ldots a_1 z^{M-1} + a_0 z^M},$$

which *is* a rational polynomial in *z*.

4. **Inverse transform of the transfer function:** For both types of transform and transfer function, the inverse transform of the transfer function yields an important time function or sequence, namely the ***impulse response***.

Continuous system: Here the impulse response, *h(t),* is the response of the system to an input impulse of infinite amplitude, zero width, and unity area that occurs at $t = 0$. In order to write the algebraic expression of the inverse LaPlace transform of a typical transfer function, you (or any other human) would need a comprehensive table of LaPlace transforms. However, your computer only needs to know one LaPlace transform pair, the one for an exponential function of time,

$$Ke^{pt} \overset{LaP}{\Leftrightarrow} \frac{K}{s-p},$$

where *K* and *p* may be complex, and the double-headed arrow stands for the LaPlace transform going to the right, and the inverse LaPlace transform going to the left.

Since any rational polynomial transfer function with *M* poles at the locations *p* in the complex plane can also be written in residue pole form,

$$H(s) = \sum_{k=1}^{M} \frac{A_k}{s - p_k},$$

this means that the impulse response of any such system in just a sum of exponential time functions,

$$h(t) = \sum_{k=1}^{M} A_k \exp[p_k t].$$

Although you may not recognize this sum of exponentials as resembling any particular continuous time function, your computer can use this sum of exponentials to calculate numerically the time response.
Sampled system: Here the impulse response sequence, h_k is the response of the system to an input sequence whose first member has unit amplitude and occurs at $t = 0$, with all other members of the input sequence being zero. In order to write the algebraic expression of the inverse z- transform of a typical transfer function, you (or any other human) would need a

comprehensive table of z transforms. However, your computer only needs to know one z transform pair, the one for an exponential function of time.

In order to see what the z-transform of an exponential sequence is, consider the samples taken each T seconds from a decreasing exponential function,

$$f_n = \exp[-bnT].$$

Since you know each element in the sequence, you can write its z-transform using the operational definition above, to get

$$F(z) = \sum_{n=0}^{\infty} \exp[-bnT]z^{-n} = \sum_{n=0}^{\infty} u^n = 1 + u + u^2 .....,$$

where we defined

$$u \equiv \exp[-bT]z^{-1}.$$

The last form of the power series above is a well-known one, and has been proved to converge to

$$\sum_{n=0}^{\infty} u^n = 1 + u + u^2 .... = \frac{1}{1-u}.$$

Therefore, the z-transform of a decreasing exponential time sequence is given in the z-transform pair,

$$A\exp[-bt] \overset{z}{\Leftrightarrow} \frac{A}{1-\exp[-bT]z^{-1}} = \frac{Az}{z-\exp[-bT]}.$$

Since the exponential decay constant b, and the sampling period T are known, the form on the right is a rational polynomial in z. Since the exponential decay constant could be complex and could have either sign, an equivalent z-transform pair, exactly analogous to the one we developed for continuous systems, would be

$$A\exp[pt] \overset{z}{\Leftrightarrow} \frac{Az}{z-\exp[pT]}.$$

Finally, let's put the impulse response sequence, h_n, into a sum-of-exponentials form analogous to the one we developed for continuous systems. Start with the rational polynomial discrete transfer function H(z), for your system, count the number of poles it has as M, divide it by z, (**which will give it one more pole at z=0**) and put the result into residue-pole form,

$$\frac{H(z)}{z} = \frac{A_0}{z} + \sum_{k=1}^{M} \frac{A_k}{z-q_k}.$$

The first term represents a new pole at z=0 introduced when we divided $H(z)$ by z, while the sum in the second term runs over the ***original*** poles at locations q_k in the complex z-plane that were present in the discrete transfer function, $H(z)$.

Next, multiply both sides by z to get back the original discrete transfer function,

$$H(z) = A_0 + \sum_{k=1}^{M} \frac{A_k z}{z-q_k}.$$

This looks like the z-transform of a sum-of-exponentials if we make the substitution, $q_k = \exp[p_k T]$. Then the z-transform of the impulse response is

$$H(z) = A_0 + \sum_{k=1}^{M} \frac{A_k z}{z - \exp[p_k T]}.$$

The first term above is just a constant. Its inverse z-transform is just a single pulse at $t = 0$. All other terms in h_k will come from the original poles, q_k, in the discrete transfer function for the system. Except for the pulse at $t = 0$, the impulse response looks like a sampled version of the time function,

$$h(t) = \sum_{k=1}^{M} A_k e^{p_k t},$$

analogous to the continuous system.

WARNING! For continuous systems, the pole locations give the decay constants in the exponentials directly. For discrete systems, this is not the case: you must make use of the relation, $q_k = \exp[p_k T]$.

5. **Stability information from pole location:** Since any impulse response can be reduced to a sum of exponentials using the residue-pole form of the transfer function, you can use pole locations to determine stability.

   Continuous system: The poles appear directly in the decay-constants of all of the exponentials. For stability, none of these exponentials can be growing, so for poles at $p = \sigma \pm j\omega$, stability requires $\sigma \le 0$, or ***no poles in the right half plane.***

   Sampled system: The poles are located at the points q_k in the complex z-plane, but the decay constants are not the same as the poles. Instead, they are related to the poles by the equation, $q_k = \exp[p_k T]$. Solving for p_k gives

   $$p_k = \frac{1}{T} \ln q_k = \frac{1}{T}\left\{ \ln|q_k| + j\angle q_k \right\}.$$

   To avoid any growing exponentials in the impulse response, the real part of p_k must be zero or negative, which means that the ***poles must lie within a circle of unit radius in the complex z-plane.***

   Note: The above mapping of complex *s*-plane locations into their corresponding complex *z*-plane can be written as

   $$q = \exp(pT) = \exp(\sigma T)\exp(j\omega T),$$

   which tells us something interesting about the consequences of not satisfying the sampling theorem in our design work. If the angle of q in the complex z-plane reaches and passes π, the mapping of the left-half of the s-plane into the inside of the unit circle in the z-plane "folds over" back onto itself. This condition corresponds to $\omega T = \pm\pi = \frac{2\pi f}{f_s}$, where f_s is the sampling frequency, which reduces to $f = f_s/2$. This frequency is the maximum frequency that can be in the system and still satisfy the sampling theorem.

6. **Transform corresponding to a time-delay of one sampling period, *T*:** The complex planes that show the real and imaginary parts of the transform variables *s* and *z* map into each other. You can see this mapping relationship by comparing the transforms of delayed time-functions or sequences.

Continuous system:

$$f(t-T)\overset{LaP}{\Leftrightarrow}F(s)e^{-sT}$$

Sampled system:

$$f(t-T)\overset{z}{\Leftrightarrow}F(z)z^{-1}$$