Supplementary Proof for "Exact and Approximate Construction of Digital Phase Modulations by Superposition of AMP" by P. A. Laurent

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***Abstract*—In this letter, we supplement the derivation in Laurent's paper by mathematically proving that combining binary continuous phase modulation (CPM) signals in different bit durations results in the decomposed CPM expression consisting of a set of continuous amplitude-modulated pulse functions.**

***Index Terms*—Amplitude-modulated pulses (AMP), continuous phase modulation (CPM).**

I. Introduction

In [1], Laurent decomposes the binary continuous phase modulation (CPM) with a noninteger modulation index into the superposition of amplitude-modulated pulses (AMP). The decomposition of M-ary CPM signals is studied in [2]. And the case of the integer-modulation index is addressed in [3] and [4]. Laurent's decomposition approach and its extensions are used by many researchers in designing reduced-complexity CPM receivers [5]–[12].

In the last step of the derivation in [1], it is shown that the summation of the expressions of the CPM signal for different bit durations results in the well-known AMP decomposed expression for the CPM signal, which is expressed in terms of a set of overlapping AMP functions [1]. This derivation step is justified using only descriptive explanations [1]. In this letter, we provide mathematical proof for this importation step of deriving the decomposed CPM expression, which makes the derivation in [1] more complete and rigorous.

This letter is organized as follows. In Section II, we summarize the derivation procedure in [1]. The supplementary proof is presented in Section III. Finally, the conclusion is given in Section IV.

II. Laurent's Derivation

In this section, we shall summarize the derivation procedure given in [1]. Though some expressions given here are slightly different from those in [1], they are mathematically equivalent. In [1], the CPM signal in the Nth bit duration is given by

$$X_N(t) = J^{A_{0,N-L}} \prod_{i=0}^{L-1} \exp j[a_{N-i}\varphi(t-(N-i)T)], \quad NT \le t \le (N+1)T \qquad (1)$$

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where $J = \exp(jh\pi)$, a_n is the nth data symbol, and

$$A_{0,N} = \sum_{n=-\infty}^{N} a_n. \qquad (2)$$

$\varphi(t)$ is the phase-shift function that satisfies the following conditions:

$$\varphi(t) = \begin{cases} 0, & t \le 0 \\ h\pi, & t \ge LT \end{cases} \qquad (3)$$

where T is the bit duration and h is known as the modulation index. Here, we define a window function with length of nT as

$$W_{nT}(t) = \begin{cases} 1, & 0 \le t \le nT \\ 0, & \text{otherwise.} \end{cases} \qquad (4)$$

By using the window function, the CPM signal, denoted as $X(t)$, for infinite transmission time can be expressed in terms of the CPM signal in the Nth bit duration $X_N(t)$ as

$$X(t) = \sum_{N=-\infty}^{\infty} X_N(t) W_T(t-NT). \qquad (5)$$

An important step of the derivation is to replace the complex exponential in (1) as follows:

$$\exp[ja_n\varphi(t-nT)] = \frac{\sin[\Phi - \varphi(t-nT)]}{\sin(\pi h)} + J^{a_n}\frac{\sin[\varphi(t-nT)]}{\sin(\pi h)} \qquad (6)$$

where $\Phi = \pi h$. The above equation requires the modulation index h to be noninteger. As mentioned earlier, the case of the integer-modulation index is addressed in [3] and [4]. In [1], as another important step of derivation, a generalized phase pulse is defined as follows:

$$\Psi(t) = \begin{cases} \varphi(t), & t < LT \\ \Phi - \varphi(t-LT), & t \ge LT. \end{cases} \qquad (7)$$

The complex exponential in (6) can be expressed in terms of $\Psi(t)$ in (7) as

$$\exp[ja_n\varphi(t-nT)] = \frac{\sin[\Psi(t-nT+LT)]}{\sin(\pi h)} + J^{a_n}\frac{\sin[\Psi(t-nT)]}{\sin(\pi h)}, \quad nT \le t \le nT + LT. \qquad (8)$$

Note that the time constraint in (8) is to ensure the validity of the substitution of the arguments of the sine functions in (6) with the generalized phase pulse $\Psi(t)$.

In [1], a basic function for interpretation is defined as

$$S_n(t) = \frac{\sin[\Psi(t+nT)]}{\sin(\pi h)}. \quad (9)$$

By substituting (8) and (9) into (1), we obtain

$$X_N(t) = J^{A_{0,N-L}} \prod_{i=0}^{L-1} \left[S_{i+L-N}(t) + J^{a_{N-i}} S_{i-N}(t)\right], \quad NT \le t \le (N+1)T. \quad (10)$$

The remaining part of the derivation in [1] continues from (10) and essentially shows that the CPM signal $X(t)$ in fact has an expression equivalent to

$$X(t) = \sum_{N=-\infty}^{\infty} J^{A_{0,N-L}} J^{a_N} S_{-N}(t) \prod_{i=1}^{L-1} [S_{i+L-N}(t) + J^{a_{N-i}} S_{i-N}(t)]. \quad (11)$$

We shall show the equivalence of (11) and the decomposed CPM expression in [1, eq. (14)]. First, we define the index $K = 0, \ldots, 2^{L-1}-1$. K can be represented using a L-bit binary variable. We denote the ith bit of the binary form of K as $K(i)$. For example, for $L = 3$ and $K = 6$, the binary form of K is 110 with $K(0) = 0, K(1) = 1, K(2) = 1$. Using this notation, and by expanding the summation term in (11), we obtain

$$X(t) = \sum_{N=-\infty}^{\infty} \sum_{K=0}^{2^{L-1}-1} J^{A_{0,N-L}+a_N+\sum_{i=1}^{L-1}[1-K(i)]a_{N-i}} \cdot S_{-N}(t) \prod_{i=1}^{L-1} S_{i-N+K(i)L}(t). \quad (12)$$

We follow [1] in defining the AMP function and the complex phase coefficient as

$$C_K(t) = S_{-N}(t) \prod_{i=1}^{L-1} S_{i-N+K(i)L}(t) \quad (13)$$

$$A_{K,N} = A_{0,N} - \sum_{i=1}^{L-1} K(i) a_{N-i} \quad (14)$$

where $K = 0, \ldots, 2^{L-1}-1$. By substituting (13) and (14) into (12), we obtain the well-known decomposed CPM expression as

$$X(t) = \sum_{N=-\infty}^{\infty} \sum_{K=0}^{2^{L-1}-1} J^{A_{K,N}} C_K(t). \quad (15)$$

Therefore, the equivalence of (11) and the decomposed CPM expression in [1, eq. (14)] is verified.

As mentioned earlier, in [1], the mathematical details of deriving (11) from (10) was substituted with only descriptive explanations. In the next section, we shall provide the mathematical derivation of (11) by showing that when substituting (10) into (5):

- how $X_N(t)$ for different N values can be combined to form the continuous AMP functions;
- how the window function $W_T(t-NT)$ can be eliminated.

III. Derivation of (11)

The CPM signal in the Nth bit duration, $X_N(t)$, contains two components. The first component is the summation of all decomposed AMP functions corresponding to the Nth bit duration, and its expression is identical to the summation term in (11). The second component contains the overlapping parts of the AMP functions corresponding to the neighboring bit durations. We separate these two components of $X_N(t)$ in (10) as

$$X_N(t) = \alpha_N(t) + \beta_N(t), \quad NT \le t \le (N+1)T \quad (16)$$

where

$$\alpha_N(t) = J^{A_{0,N-L}} J^{a_N} S_{-N}(t) \prod_{i=1}^{L-1} [S_{i+L-N}(t) + J^{a_{N-i}} S_{i-N}(t)] \quad (17)$$

$$\beta_N(t) = J^{A_{0,N-L}} S_{L-N}(t) \prod_{i=1}^{L-1} [S_{i+L-N}(t) + J^{a_{N-i}} S_{i-N}(t)]. \quad (18)$$

By substituting (16) into (5), we obtain

$$X(t) = \sum_{N=-\infty}^{\infty} \alpha_N(t) W_T(t-NT) + \beta_N(t) W_T(t-NT). \quad (19)$$

In the time duration $NT \le t \le (N+1)T$, $\beta_N(t)$ in (18) is the summation of L time-shifted copies of $\alpha_N(t)$ in (17). We shall prove this property with the following lemma.

Lemma 1: With $\alpha_N(t)$, $\beta_N(t)$, and $W_T(t)$ defined in (17), (18), and (4), respectively, the following equation is true:

$$\beta_N(t) W_T(t-NT) = \sum_{m=1}^{L} \alpha_{N-m}(t) W_T(t-NT). \quad (20)$$

Proof: First, let us obtain the expression for $\alpha_{N-m}(t) W_T(t-NT)$, where $\alpha_{N-m}(t)$ is equal to $\alpha_N(t)$ shifted by mT to the left, and can be obtained from (17) by replacing N with $N-m$. From (3), (7), and (9), it follows that $S_n(t)$ is nonzero only for $-nT \le t \le 2LT - nT$. Therefore, (21) holds for $n \ge 2L$ or $n \le -1$ as shown in (21)–(24) at the bottom of the next page. Using (21) and (2), and after some manipulation, we get the expression in (22). Using (22), we prove in the Appendix the hypothesis $\mathcal{H}(\lambda)$ in (23), using the method of induction. From (23), let $\lambda = 1$, and we have (24). By substituting (18) into (24), (20) follows. ■

We shall prove another lemma which is also required in obtaining the decomposed CPM expression in (11).

Lemma 2: $\alpha_N(t)$ defined in (17) is nonzero only in the time duration from NT to $(L+1)T + NT$. That is to say

$$\alpha_N(T) = \alpha_N(t) W_{(L+1)T}(t-NT). \quad (25)$$

Proof: As mentioned earlier, $S_n(t)$ is nonzero for $-nT \leq t \leq 2LT - nT$. Thus

$$S_n(t) = S_n(t)W_{2LT}(t - nT). \quad (26)$$

Similarly, we can also show that

$$S_{n+L}(t) + J^{a_{-n}}S_n(t) = [S_{n+L}(t) + J^{a_{-n}}S_n(t)] \cdot W_{3LT}(t - nT + LT). \quad (27)$$

We can substitute (26) and (27) into (17) to obtain

$$\alpha_N(t) = \alpha_N(t)\cdot\left[W_{2LT}(t - NT)\prod_{i=1}^{L-1} W_{3LT}(t - NT + LT + iT)\right]. \quad (28)$$

We can observe that the overlapping part of the window functions in (28) is actually $W_{(L+1)T}(t - NT)$. Hence

$$W_{(L+1)T}(t - NT) = W_{2LT}(t - NT)\prod_{i=1}^{L-1} W_{3LT}(t - (N - L - i)T). \quad (29)$$

Substitute (29) into (28), and *Lemma 2* is proved. ■

Finally, we derive (11) by proving the following proposition.

Proposition 1: With $X(t)$ and $\alpha_N(t)$ defined in (5) and (17), respectively, the received CPM signal $X(t)$ is given as

$$X(t) = \sum_{N=-\infty}^{\infty} \alpha_N(t). \quad (30)$$

Proof: Using *Lemma 1*, by substituting (20) into (19), we can obtain

$$X(t) = \sum_{N=-\infty}^{\infty}\sum_{m=0}^{L} \alpha_{N-m}(t)W_T(t - NT). \quad (31)$$

By changing summation index, (31) can be re-expressed as

$$X(t) = \sum_{N=-\infty}^{\infty} \alpha_N(t)\sum_{m=0}^{L} W_T(t - NT - mT) = \sum_{N=-\infty}^{\infty} \alpha_N(t)W_{(L+1)T}(t - NT). \quad (32)$$

Using *Lemma 2*, we can substitute (25) into (32) and complete the proof for *Proposition 1*. ■

IV. Conclusion

We supplement the derivation in [1] by mathematically deriving the decomposed CPM expression in (11).

Appendix
Proof of $\mathcal{H}(\lambda)$

1) Proof of $\mathcal{H}(L-2)$

From (22), we can obtain

$$\sum_{m=L-2}^{L} \alpha_{N-m}(t)W_T(t - NT) = J^{A_{0,N-L}}\prod_{i=0}^{L-3} S_{i+L-N}(t) \times \prod_{i=L-2}^{L-1}[S_{i+L-N}(t) + J^{a_{N-i}}S_{i-N}(t)] \times W_T(t - NT). \quad (33)$$

Therefore, $\mathcal{H}(L-2)$ is proved.

2) Proof of $\mathcal{H}(\lambda - 1)$ given $\mathcal{H}(\lambda)$

$\mathcal{H}(\lambda - 1)$ corresponds to

$$\sum_{m=\lambda-1}^{L} \alpha_{N-m}(t)W_T(t - NT) = \sum_{m=\lambda}^{L}\alpha_{N-m}(t)W_T(t-NT)+\alpha_{N-\lambda+1}(t)W_T(t-NT). \quad (34)$$

$$S_{n-N}(t)W_T(t - NT) = 0 \quad \text{for } n \geq 2L \quad \text{or} \quad n \leq -1 \quad (21)$$

$$\alpha_{N-m}(t)W_T(t - NT) = \begin{cases} J^{A_{0,N-L}}J^{a_{N-m}}S_{m-N}(t)\prod_{i=0}^{m-1} S_{i+L-N}(t)\prod_{i=1+m}^{L-1}[S_{i+L-N}(t) \\ \quad + J^{a_{N-i}}S_{i-N}(t)]\, W_T(t - NT), & 1 \leq m \leq L-2 \\ J^{A_{0,m}}\prod_{i=0}^{L-1} S_{i-N+m}(t)W_T(t - NT), & L-1 \leq m \leq L \end{cases} \quad (22)$$

$$\mathcal{H}(\lambda) : \sum_{m=\lambda}^{L} \alpha_{N-m}(t)W_T(t - NT) = J^{A_{0,N-L}}\prod_{i=0}^{\lambda-1} S_{i+L-N}(t)\prod_{i=\lambda}^{L-1}[S_{i+L-N}(t) + J^{a_{N-i}}S_{i-N}(t)] \cdot W_T(t - NT), \quad 1 \leq \lambda \leq L-2 \quad (23)$$

$$\sum_{m=1}^{L} \alpha_{N-m}(t)W_T(t - NT) = J^{A_{0,N-L}}S_{L-N}(t)\prod_{i=1}^{L-1}[S_{i+L-N}(t) + J^{a_{N-i}}S_{i-N}(t)]\, W_T(t - NT) \quad (24)$$

Since $\mathcal{H}(\lambda)$ is assumed to be true, we can substitute (23) and (22) with $m = \lambda - 1$ into (34) as in

$$\begin{aligned}
\sum_{m=\lambda-1}^{L} & \alpha_{N-m}(t) W_T(t-NT) \\
&= J^{A_{0,N-L}} \prod_{i=0}^{\lambda-1} S_{i+L-N}(t) \\
&\quad \times \prod_{i=\lambda}^{L-1} \left[S_{i+L-N}(t) + J^{a_{N-i}} S_{i-N}(t)\right] \\
&\quad \times W_T(t-NT) + J^{A_{0,N-L}} \\
&\quad \times \prod_{i=0}^{\lambda-2} S_{i+L-N}(t) \prod_{i=\lambda}^{L-1} \left[S_{i+L-N}(t)\right. \\
&\quad \left. + J^{a_{N-i}} S_{i-N}(t)\right] W_T(t-NT) J^{a_{N-\lambda+1}} S_{\lambda-1-N}(t) \\
&= J^{A_{0,N-L}} \prod_{i=0}^{\lambda-2} S_{i+L-N}(t) \\
&\quad \times \prod_{i=\lambda-1}^{L-1} \left[S_{i+L-N}(t) + J^{a_{N-i}} S_{i-N}(t)\right] \\
&\quad \times W_T(t-NT).
\end{aligned} \tag{35}$$

Therefore, $\mathcal{H}(\lambda - 1)$ is also true, given that $\mathcal{H}(\lambda)$ is true. This completes the proof for the hypothesis $\mathcal{H}(\lambda)$. □

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